# Exercises for 'Topics in complex analysis'

(29/10/2025)

**H 7.1 (Holomorphic interpolation)**
Let $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ be a closed discrete set and $\{b_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ be an arbitrary countable set. Show that there exists a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ such that $f(a_n) = b_n$ for all $n \in \mathbb{N}$.
**Hint**: Combine the Weierstrass product theorem with the Mittag-Leffler theorem for a suitable sequence of principal parts.

**H 7.2 (Representation of meromorphic functions as quotients)**
Let $h : \mathbb{C} \to \mathbb{C}$ be a meromorphic function (i.e. its singularities are isolated and only poles of finite order). Show that there exist two entire functions $f, g : \mathbb{C} \to \mathbb{C}$ with no common zeros such that $h = \frac{f}{g}$.

**H 7.3 (Weierstrass product theorem on open sets)**
Let $U \subset \mathbb{C}$ be an open set and let $\{a_n\}_{n\in\mathbb{N}} \subset U$ be a sequence with no accumulation points in $U$. Set $o_n := \#\{k \in \mathbb{N} : a_k = a_n\}$. We claim that there exists a holomorphic function $f : U \to \mathbb{C}$ such that $Z(f) = \{a_n\}_{n\in\mathbb{N}}$ and $o_{a_n}(f) = o_n$ for all $n \in \mathbb{N}$. Moreover, as we shall see in the proof, the function $f$ can be taken as an infinite product.
The argument splits into two steps. Similary to the Mittag-Leffler theorem, we recenter some of the Weierstrass factors and replace $E_n\big(\frac{z}{a_n}\big)$ by $E_n\big(\frac{a_n - c_n}{z - c_n}\big)$, for a suitable sequence $c_n$. We denote by $S'$ the set of accumulation points (in $\mathbb{C}$) of the sequence $\{a_n\}_{n\in\mathbb{N}}$. If $S' = \emptyset$ there is nothing to prove as we can apply Theorem 4.2. Hence assume that $S' \neq \emptyset$.

a) Suppose that there exists a sequence $\{c_n\}_{n\in\mathbb{N}} \subset S'$ such that $\lim\limits_{n\to+\infty} |a_n - c_n| = 0$. Show that the infinite product

$$f(z) = \prod_{n=1}^{\infty} E_n\Big(\frac{a_n - c_n}{z - c_n}\Big)$$

converges locally normally on $U$ and satisfies $Z(f) = \{a_n\}_{n\in\mathbb{N}}$ and $o_{a_n}(f) = o_n$ for all $n \in \mathbb{N}$.

b) Split the set $S := \{a_n\}_{n\in\mathbb{N}}$ as in Lemma 2.7 and use Lemma 2.8 to conclude the proof by combining a) and Theorem 4.2.

**H 7.4 (Blaschke condition)**
Prove that if $f : B_1(0) \to \mathbb{C}$ is holomorphic, bounded, and not identically zero, and $z_1, z_2, \ldots$ are its zeros (with $|z_n| < 1$ and listed with multiplicity), then

$$\sum_n (1 - |z_n|) < \infty.$$

**Hint**: Use Jensen's formula.

**H 7.5 (Blaschke products)**
In this problem, we discuss Blaschke products, which are bounded analogues in the disc of the Weierstrass products for entire functions.

a) Show that for any $0 < |\alpha| < 1$ and $|z| \leq r < 1$, we have the inequalities

$$\left|\frac{\alpha + |\alpha| z}{(1-\overline{\alpha} z)\alpha}\right| \leq \frac{1+r}{1-r} \qquad \text{and} \qquad \left|\frac{\alpha - z}{1-\overline{\alpha} z}\right| \leq 1.$$

b) Let $\{\alpha_n\}_n$ be a sequence in $B_1(0)$ such that $\alpha_n \neq 0$ for all $n$ and

$$\sum_n (1 - |\alpha_n|) < \infty.$$

Note that this will be the case if $\{\alpha_n\}_n$ are the zeros of a bounded holomorphic function on the unit disc (see Exercise H 7.4). Show that the product

$$f(z) = \prod_n \frac{\alpha_n - z}{1 - \overline{\alpha_n} z} \frac{|\alpha_n|}{\alpha_n}$$

converges uniformly for $|z| \leq r < 1$, hence defines a holomorphic function on the unit disc having precisely the zeros $\alpha_n$ and no other zeros. Show also that $|f(z)| \leq 1$.